A CONTINUITY THEOREM FOR FUCHSIAN GROUPS[1]

BY

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Abstract. On a given Riemann surface, fix a discrete (finite or infinite) sequence of points $\{P_k\}$, $k = 1, 2, 3, \ldots$, and associate to each P_k an "integer" ν_k (which may be $1, 2, 3, \ldots$, or ∞). This sequence of points and "integers" is called a "signature" on the Riemann surface. With only a few exceptions, a Riemann surface with signature can always be represented by a Fuchsian group. We investigate here the dependence of the group on the number ν_k. More precisely, keeping the points P_k fixed, we vary the numbers ν_k in such a way that the signature tends to a limit signature. We shall prove that the corresponding representing Fuchsian group converges to the Fuchsian group which corresponds to the limit signature.

I. **Introduction.** On a given Riemann surface, fix a discrete (finite or infinite) sequence of points $\{P_k\}$, $k = 1, 2, 3, \ldots$, and associate to each P_k an "integer" ν_k (which may be $1, 2, 3, \ldots$, or ∞). This sequence of points and "integers" is called a "signature" on the Riemann surface. With only a few exceptions, a Riemann surface with signature can always be represented by a Fuchsian group. This is the celebrated limit circle theorem of Koebe. We investigate here the dependence of the group on the numbers ν_k. More precisely, keeping the points P_k fixed we vary the numbers ν_k in such a way that the signature tends to a limit signature. We shall prove that *the corresponding representing Fuchsian group converges to the Fuchsian group which corresponds to the limit signature*, in a sense which we shall specify in the text.

This main result is established in §VII.

II. **Preliminaries.** In the first part of this section we state the limit circle theorem of Koebe [10]. The rest of this section is devoted to the statement and proof of some properties of the solution of an elliptic partial differential equation.

DEFINITION 1. Let S be a Riemann surface and $\{P_k\}$, $k = 1, 2, 3, \ldots$, be a discrete (finite or infinite) sequence of points on S. Let there be an "integer" $\nu_k \geq 2$ associated with each point P_k (here ν_k may be an actual integer, or the symbol ∞). The following two cases are excluded: (i) $S = \hat{C} = C \cup \{\infty\}$, the Riemann sphere, with one point P and the associated "integer" $\nu \neq \infty$ and (ii) $S = \hat{C}$, with two points P_1, P_2

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and $\nu_1 \neq \nu_2$. The triple $(S, \{P_k\}, \{\nu_k\})$ is called a *Riemann surface with signature*. For simplicity, we shall write P, ν, (S, P, ν) for $\{P_k\}$, $\{\nu_k\}$, $(S, \{P_k\}, \{\nu_k\})$ respectively.

THEOREM 1 (KOEBE). *Let (S, P, ν) be a Riemann surface with signature. Let* $S^{\#} = S - \bigcup_{\nu_k = \infty} \{P_k\}$, $S^{b} = S - \bigcup_{\nu_k \geq 2} \{P_k\}$.

Then there exists a simply connected Riemann surface $V^{\#}$, a properly discontinuous group G of conformal selfmappings of $V^{\#}$, and a conformal mapping from $V^{\#}/G$ onto $S^{\#}$ such that

(i) *under this mapping S^{b} is conformally equivalent to V^{b}/G, where V^{b} is $V^{\#}$ with all fixed points of elements of G removed, and*

(ii) *the natural projection $V^{\#} \to V^{\#}/G$ followed by the conformal mapping $V^{\#}/G \to S^{\#}$ is locally 1-1 at each point of V^{b} and is ν_k to 1 at the preimages of P_k with $\nu_k < \infty$.*

The group G is determined uniquely up to conjugation in the full group of conformal selfmappings of $V^{\#}$.[2]

REMARK. The fact that $V^{\#}$ is simply connected implies that $V^{\#}$ is conformally equivalent to $\hat{C}$, C or Δ, where Δ denotes, here and hereafter, the unit disk. We can actually enumerate all Riemann surfaces with signature (S, P, ν) such that $V^{\#}$ is conformally equivalent to $\hat{C}$ or C as follows. (See Ford [7, Chapter IX, §§94, 95].)

(i) $S = \hat{C}$ with one point P, $\nu = \infty$,

(ii) $S = \hat{C}$ with two points P_1, P_2, and $\nu_1 = \nu_2$,

(iii) $S = \hat{C}$ with three points P_1, P_2, P_3, and $1/\nu_1 + 1/\nu_2 + 1/\nu_3 \geq 1$,

(iv) $S = \hat{C}$ with four points P_1, P_2, P_3, P_4 and $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 2$,

(v) $S = $ a torus and there are no points P_k.

We shall call these cases *exceptional*.

COROLLARY. *Given a nonexceptional Riemann surface with signature (S, P, ν), there exists a Fuchsian group G with the unit circle as its fixed circle such that $S^{\#} = S - \bigcup_{\nu_k = \infty} \{P_k\}$ is conformally equivalent to Δ/G. $S^{b} = S - \bigcup_{\nu_k \geq 2} \{P_k\}$ is conformally equivalent to Δ_G/G, where $\Delta_G = \Delta - \{all elliptic fixed points of G\}$, and S is conformally equivalent to $\hat{\Delta}_G/G$, where $\hat{\Delta}_G = \Delta \cup \{all parabolic fixed points of G\}$. The natural projection $\Delta \to \Delta/G$ followed by the conformal mapping $\Delta/G \to S^{\#}$ is locally 1-1 at each point of Δ_G, and is ν_k to 1 at the preimages of P_k with $\nu_k < \infty$.*

The Fuchsian group G is determined uniquely up to conjugation by a Möbius transformation.

Proof. We have only to show that S is conformally equivalent to $\hat{\Delta}_G/G$. But this is exactly a result of Heins [8].

In this case we say that (S, P, ν) is *represented* by a Fuchsian group G. We shall also allow $\nu_k = 1$. That is, given a Riemann surface S and a discrete (finite or

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([2]) A complete proof of the limit circle theorem of Koebe can be found in the above-mentioned dissertation.

infinite) sequence of points $\{P_k\}$, $k = 1, 2, 3, \ldots$, on S, we associate an "integer" ν_k to each point P_k, where $\nu_k \geqq 1$ or $= \infty$.

DEFINITION 2. (i) A triple (S, P, ν) is called a Riemann surface with signature if after ignoring all points P_k with $\nu_k = 1$, the resulting triple is a Riemann surface with signature as in Definition 1.

(ii) A triple (S, P, ν) is called exceptional if after ignoring all points P_k with $\nu_k = 1$, the resulting triple is an exceptional Riemann surface with signature, as defined above.

We shall discuss only nonexceptional triples throughout this paper.

The rest of this section will be devoted to the study of some properties of real-valued solutions of the elliptic partial differential equation

$$(1) \qquad (\partial^2 / \partial t\, \partial \bar{t}) \log \lambda(t) = \lambda^2(t)$$

or, equivalently,

$$(2) \qquad (\partial^2 / \partial t\, \partial \bar{t}) u(t) = e^{2u(t)},$$

where $\lambda(t)$, $u(t)$ are real-valued functions of complex-variable t, and $\lambda = e^u$. From now on whenever we say that u satisfies (2) we mean that u is a real-valued C^2 solution of (2). The proof of Lemmas 1, 2, 3 and 4 can be accomplished by standard potential-theoretical arguments as in [2] and is therefore omitted.

LEMMA 1. *If u_1, u_2 satisfy (2) in some domain D, then $u_1 - u_2$ has neither positive maxima nor negative minima in D.*

LEMMA 2. *If $\{u_i\}$, $i = 1, 2, 3, \ldots$, satisfy (2) in $|t| < R$, and for every i, $|u_i| \leqq M$ there, then in $|t| < R_0 < R$, any infinite sequence of the family $\{u_i\}$ has a uniformly convergent subsequence.*

LEMMA 3. *If $\{u_i\}$, $i = 1, 2, 3, \ldots$, satisfy (2) in $|t| < R$ and $u_i \rightarrow u$ uniformly in $|t| < R$ as $i \rightarrow \infty$, then u is of class C^2 and satisfies (2) in $|t| < R_0 < R$.*

COROLLARY. *If $\{u_i\}$, $i = 1, 2, 3, \ldots$, satisfy (2) in D and $u_i \rightarrow u$ normally in D as $i \rightarrow \infty$, then u is of class C^2 and satisfy (2) in D.*

LEMMA 4. *If u satisfies (2) in $0 < |t| < R$ and u is bounded, then u is of class C^2 in $|t| < R$ and satisfies (2) there.*

LEMMA 5. *If $u(t)$ satisfies (2) in $0 < |t| < R$ and $u(t) \sim b + (1/n - 1) \log |t|$, as $t \rightarrow 0$, where b is a constant and n is a positive integer, then there exists a function $v(z)$, $|z| < R^{1/n}$, such that $v(z)$ is of class C^2, satisfying (2), $v(e^{2\pi i/n}z) = v(z)$ for $|z| < R^{1/n}$ and $v(z) = u(z^n) + \log n + (n-1) \log |z|$ for $0 < |z| < R^{1/n}$.*

Proof. Set $z = t^{1/n}$. Define $v(z) = u(z^n) + \log n + (n - 1) \log |z|$ for $0 < |z| < R^{1/n}$. Then clearly $v(z)$ is well defined and $v(e^{2\pi i/n}z) = v(z)$. By assumption, $u(z^n) \sim b + (1 - n) \log |z|$, as $z \rightarrow 0$. Hence $v(z) \sim b + \log n$ as $z \rightarrow 0$. Therefore $v(z)$ is bounded

in $0 < |z| < R^{1/n}$. But $v(z)$ is of class C^2 in $0 < |z| < R^{1/n}$, and satisfies (2) there; by Lemma 4, $v(z)$ is of class C^2 in $|z| < R^{1/n}$ and satisfies (2) there.

COROLLARY. *If $\lambda(t)$ is a positive C^2 solution of* (1) *in $0 < |t| < R$ and $\lambda(t) \sim a|t|^{1/n-1}$ as $t \to 0$, where $a > 0$ is a constant and n is a positive integer, then there exists a function $\mu(z)$, $|z| < R^{1/n}$, such that $\mu(z)$ is positive, of class C^2, satisfying* (1), *$\mu(e^{2\pi i/n}z) = \mu(z)$ for $|z| < R^{1/n}$ and $\mu(z) = \lambda(z^n)n|z|^{n-1}$ for $0 < |z| < R^{1/n}$.*

LEMMA 6. *If $\{u_i\}$, $i = 1, 2, 3, \ldots$, $\{\hat{u}_j\}$, $j = 1, 2$, satisfy* (2) *in $0 < |t| < R$, $\hat{u}_1 \leqq u_i \leqq \hat{u}_2$ in $0 < |t| < R$ for all i, $u_i(t) \sim b_i + (1/n - 1) \log|t|$, $\hat{u}_j(t) \sim \hat{b}_j + (1/n - 1) \log|t|$, as $t \to 0$, where b_i, $\hat{b}_j$ are constants, n is a positive integer, and finally, $u_i \to u$ normally in $0 < |t| < R$ as $i \to \infty$, then $u(t) \sim b + (1/n - 1) \log|t|$, as $t \to 0$, where b is a constant.*

Proof. By Lemma 5, there exists $\{v_i\}$, $i = 1, 2, 3, \ldots$, such that $v_i(z)$ is defined in $|z| < R^{1/n}$, of class C^2, satisfying (2) there and $v_i(z) = u_i(z^n) + \log n + (n-1) \log|z|$ in $0 < |z| < R^{1/n}$. Similarly, there exists $\hat{v}_j$, such that $\hat{v}_j(z)$ is defined in $|z| < R^{1/n}$, of class C^2, satisfying (2) there and $\hat{v}_j(z) = \hat{u}_j(z^n) + \log n + (n-1) \log|z|$ in $0 < |z| < R^{1/n}$.

By assumption, $u_i \to u$ normally in $0 < |t| < R$ as $i \to \infty$. It follows that $v_i \to v$ normally in $0 < |z| < R^{1/n}$, where $v(z) = u(z^n) + \log n + (n-1) \log|z|$. By Lemma 3, v is of class C^2 and satisfies (2) in $0 < |z| < R^{1/n}$. But $\hat{u}_1 \leqq u_i \leqq \hat{u}_2$ in $0 < |t| < R$ implies that $\hat{v}_1 \leqq v_i \leqq \hat{v}_2$ in $0 < |z| < R^{1/n}$, so that $\hat{v}_1 \leqq v \leqq \hat{v}_2$ in $0 < |z| < R^{1/n}$. Since $\hat{v}_j$ is of class C^2 in $|z| < R^{1/n}$, v is bounded in $0 < |z| < R' < R^{1/n}$. By Lemma 4, v is of class C^2 in $|z| < R'$, hence in $|z| < R^{1/n}$.

From $v(z) = u(z^n) + \log n + (n-1) \log|z|$, we conclude that $u(t) \sim v(0) - \log n + (1/n - 1) \log|t| = b + (1/n - 1) \log|t|$, as $t \to 0$, where b is a constant.

COROLLARY. *If $\{\lambda_i\}$, $i = 1, 2, 3, \ldots$, $\{\hat{\lambda}_j\}$, $j = 1, 2$, are positive C^2 solutions of* (1) *in $0 < |t| < R$, $\hat{\lambda}_1 \leqq \lambda_i \leqq \hat{\lambda}_2$ in $0 < |t| < R$ for all i, $\lambda_i(t) \sim a_i|t|^{1/n-1}$, $\hat{\lambda}_j \sim \hat{a}_j|t|^{1/n-1}$, as $t \to 0$, where a_i, $\hat{a}_j > 0$, n a positive integer and, finally, $\lambda_i \to \lambda$ normally in $0 < |t| < R$ as $i \to \infty$, then $\lambda(t) \sim a|t|^{1/n-1}$ as $t \to 0$, where $a > 0$ is a constant.*

III. **The uniqueness of the Poincaré metric.** This section consists of two parts. In the first we describe the behavior on a Riemann surface with signature, of the metric which is induced from the Poincaré metric on the unit disk. In the second part we show that this metric is uniquely determined by this behavior.

Let (S, P, ν) be a nonexceptional Riemann surface with signature. Let it be represented by a Fuchsian group G. Let $g: \hat{\Delta}_G \to S$ be the natural projection.

The Poincaré metric on Δ is defined by the element of arc length $ds = \lambda(z)|dz| = (1/(1 - |z|^2))|dz|$, $z \in \Delta$. It has constant Gaussian curvature -4, and Δ together with this metric is a complete metric space. It can be transferred to S^b. Indeed, $g: \Delta \to S^{\#}$ is locally 1-1 at each point of Δ_G, and we can define a metric on S^b by the element of arc length $\lambda_S(t)|dt| = \lambda(z)|dz|$, with $t = g(z)$, where t is a local parameter on S. This Poincaré metric on S^b is independent of the choice of the representing Fuchsian group G of (S, P, ν). Also, from the conformal invariance

of the Poincaré metric, we may use the upper half-plane U instead of Δ to describe this Poincaré metric on S^b.

Since g is a local isometry from Δ_G onto S^b, the Poincaré metric on S^b has Gaussian curvature -4. Since the Gaussian curvature is given by $K = -(4/\lambda_S^2(t))(\partial^2/\partial t \, \partial \bar{t}) \log \lambda_S(t)$, it follows that $\lambda_S(t)$ satisfies (1).

We shall next study the behavior of the Poincaré metric near points P_k with $2 \le \nu_k$.

(i) *Near a point P_k with $2 \le \nu_k < \infty$.* Let P_k be the image of a point $z_0 \in \Delta$, then z_0 is an elliptic fixed point of order ν_k. We may assume that $z_0 = 0$. In a neighborhood of P_k, there is a local parameter t (P_k corresponds to $t = 0$) on S such that $t = g(z)$ can be expressed as $t = z^{\nu_k}$ in a neighborhood of $z_0 = 0$. It follows that $\lambda_S(t) \sim b|t|^{1/\nu_k - 1}$, as $t \to 0$, where b is a positive constant depending on the choice of the local parameter t. If we use any other local parameter $\hat{t}$ on S which is 0 for $t = 0$, we have $\hat{\lambda}_S(\hat{t}) \sim \hat{b}|\hat{t}|^{1/\nu_k - 1}$, as $\hat{t} \to 0$, where $\hat{\lambda}_S(\hat{t})|d\hat{t}| = \lambda_S(t)|dt|$, and $\hat{b}$ is a positive constant.

(ii) *Near a point P_k with $\nu_k = \infty$.* Consider U instead of Δ. Let P_k be the image of a point z_0. Then z_0 is a parabolic fixed point. We may assume that $z_0 = \infty$. Then there is a local parameter t ($t = 0$ at P_k) such that $t = g(z) = \exp(2\pi i z/d)$, $d > 0$, in a neighborhood of $z_0 = \infty$. Note that the Poincaré metric on U is given by $\lambda(z)|dz| = (i/(z - \bar{z}))|dz|$, $z \in U$. Hence in a neighborhood of P_k, $\lambda_S(t) \sim -1/(2|t| \log |t|)$, as $t \to 0$. If we use another local parameter $\hat{t}$ on S which is 0 for $t = 0$, we have again $\hat{\lambda}_S(\hat{t}) \sim -1/(2|\hat{t}| \log |\hat{t}|)$, as $\hat{t} \to 0$, where $\hat{\lambda}_S(\hat{t})|d\hat{t}| = \lambda_S(t)|dt|$.

In the Poincaré metric on Δ, any (sufficiently smooth) curve leading to the boundary has infinite length. g preserves arc length. Therefore any (sufficiently smooth) curve beginning at a point on S other than points P_k with $\nu_k = \infty$ and leading to a point P_k with $\nu_k = \infty$ or to the ideal boundary of S has infinite length (measured by integrating $\lambda_S(t)|dt|$). The definition and properties of the ideal boundary can be found in Ahlfors and Sario [1, Chapter I, §13 A].

From now on, by abuse of language, we shall not distinguish between the arc length element and the metric induced by it.

The following lemma is a well-known result.

LEMMA 7. (i) *Let $\lambda(z)|dz|$ be the Poincaré metric on Δ.*

(ii) *Let $\sigma(z)|dz|$ be another metric defined on Δ, with $\sigma(z)$ of class C^2, and with Gaussian curvature -4. Assume also that Δ with the metric $\sigma(z)|dz|$ is complete.*

Then $\sigma(z) = \lambda(z)$.

THEOREM 2. *Let (S, P, ν) be a nonexceptional Riemann surface with signature. Let it be represented by a Fuchsian group G. Let $\lambda_S(t)|dt|$ be the Poincaré metric on S^b defined by G. Let $\sigma_S(t)|dt|$ be another metric defined on S^b, with $\sigma_S(t)$ of class C^2. At any point P_k with $2 \le \nu_k < \infty$, and for all local parameter t with $t = 0$ at P_k, let $\sigma_S(t) \sim d|t|^{1/\nu_k - 1}$ as $t \to 0$, where d is a positive constant depending on P_k and on the choice of the local parameter t. Let the metric $\sigma_S(t)|dt|$ have constant Gaussian*

curvature -4. Finally, assume that in the metric $\sigma_S(t)|dt|$ any (sufficiently smooth) curve on S beginning at a point other than P_k with $\nu_k=\infty$ and leading to a point P_k with $\nu_k=\infty$ or to the ideal boundary of S has infinite length.

Then the two metrics coincide: $\sigma_S(t)=\lambda_S(t)$, where t is any local parameter.

Proof. Let $g: \Delta \rightarrow S^\#$ be the natural projection. By assumption $\lambda_S(t)|dt| = \lambda(z)|dz|$, $t=g(z)$, where $\lambda(z)|dz|$ is the Poincaré metric on Δ. Lift $\sigma_S(t)|dt|$ to a metric $\sigma(z)|dz|$ on Δ_G by g, so that $\sigma_S(t)|dt|=\sigma(z)|dz|$, $t=g(z)$. Hence $\sigma_S(t)/\lambda_S(t) =\sigma(z)/\lambda(z)$, $t=g(z)$. By assumption, at any point P_k with $2 \leqq \nu_k < \infty$, and in any local parameter t with $t=0$ at P_k, we have $\lim_{t \to 0} (\sigma_S(t)/\lambda_S(t))=c$, where c is a positive constant depending on the point P_k only.

Also $\sigma(z)$ is of class C^2 in Δ_G. By what was just said, at an elliptic fixed point z_0 of G in Δ, we have $\lim_{z \to z_0} (\sigma(z)/\lambda(z))=c$, where c is a positive constant depending on the point z_0. Hence $\sigma(z)$ is continuous and positive in Δ.

Since $\sigma_S(t)|dt|$ has Gaussian curvature -4 on S^b, $\sigma(z)|dz|$ has Gaussian curvature -4 on Δ_G. Therefore $\sigma(z)$ satisfies (1) in Δ_G. Consider a relatively compact domain D in Δ containing an elliptic fixed point z_0 of G and no others in its interior. Since $\sigma(z)$ is continuous and positive in Δ, $\log \sigma(z)$ is continuous in Δ and hence bounded in D. By Lemma 4, $\log \sigma(z)$, hence $\sigma(z)$, is of class C^2 in D and $\sigma(z)$ satisfies (1) in D. The same argument applies to all other elliptic fixed points in Δ, so that $\sigma(z)$ is of class C^2 in Δ and satisfies (1) in Δ. It follows that the metric $\sigma(z)|dz|$ is defined on Δ and has constant Gaussian curvature -4.

Since g preserves arc length, it follows from the assumption that any (sufficiently smooth) curve from an interior point to the boundary of Δ in $\sigma(z)|dz|$ has infinite length. Therefore, Δ together with the metric $\sigma(z)|dz|$ is a complete metric space. By Lemma 7, $\sigma(z)=\lambda(z)$ on Δ. This implies the conclusion of the theorem.

IV. **The approximation theorem for the metric.** Given a nonexceptional Riemann surface with signature (S, P, ν), each subdomain of S may be considered, in an obvious way, as a Riemann surface with signature. We can choose a sequence $\{S_i\}$, $i=1, 2, 3, \ldots$, of subdomains of S with the following properties: (i) $S_1 \subset\subset S_2 \subset\subset \cdots \subset\subset S$, where $\subset\subset$ means "relatively compact", (ii) $S=\bigcup_{i=1}^{\infty} S_i$, (iii) each S_i is a Riemann surface of finite type and contains at most a finite number of points P_k, (iv) the boundary of each S_i consists of a finite number of simple closed analytic curves which do not pass through any of the points P_k. For i large, the Riemann surface with signature (S_i, P, ν) is nonexceptional; here and hereafter (S_i, P, ν) means S_i together with those points $P_k \in S_i$ and their associated "integers" ν_k.

We assume now that (S_i, P, ν) is nonexceptional for $i=1, 2, 3, \ldots$. We can then represent (S_i, P, ν) by a Fuchsian group G_i, and S_i carries a Poincaré metric defined by G_i. Denote it by $\lambda_i(t)|dt|$. On the other hand, we shall write $\lambda(t)|dt|$ for $\lambda_S(t)|dt|$, the Poincaré metric on S induced by the group G which represents (S, P, ν).

THEOREM 3. *Under the circumstances described above, we have*

(i) $\lambda_i(t) \geqq \lambda_{i+1}(t)$, $\lambda_i(t) \geqq \lambda(t)$ *on* $S_i^b = S_i - \bigcup_{\nu_k \geqq 2} \{P_k\}$, *and*

(ii) $\lambda_i(t) \to \lambda(t)$ *normally on* $S^b = S - \bigcup_{\nu_k \geqq 2} \{P_k\}$, *as* $i \to \infty$.

(*Here t is the same local parameter for all λ_i and λ.*)

Proof. (i) Recall that λ and λ_i satisfy (1).

On S_i^b, set

$$\delta(t) = u_{i+1}(t) - u_i(t) = \log \lambda_{i+1}(t) - \log \lambda_i(t) = \log (\lambda_{i+1}(t)/\lambda_i(t)).$$

Note that $\delta(t)$ is invariant under local parameter change, hence δ is a function on S_i^b. In any parametric disk, both u_{i+1} and u_i satisfy (2), implying that δ has neither positive maxima nor negative minima in the neighborhood considered (Lemma 1), and hence anywhere on S_i^b.

We shall now study the behavior of δ at the points P_k with $\nu_k \geqq 2$ on S_i. We claim that δ is continuous at any point P_k with $2 \leqq \nu_k < \infty$ and has neither a positive maximum nor a negative minimum there. Recall that $\lambda_{i+1}(t) \sim b_1 |t|^{1/\nu_k - 1}$, $\lambda_i(t) \sim b_0 |t|^{1/\nu_k - 1}$, as $t \to 0$ (as usual, P_k corresponds to $t = 0$), where b_1, b_0 are positive constants depending on the choice of the local parameter. Therefore $\delta = \log (\lambda_{i+1}(t)/\lambda_i(t))$ is continuous at P_k. By the corollary of Lemma 5, there exist functions $\hat{\lambda}_{i+1}(z)$, $\hat{\lambda}_i(z)$, of class C^2, satisfying (1) in a neighborhood of $z = 0$, and $\hat{\lambda}_{i+1}(z)|dz| = \lambda_{i+1}(t)|dt|$, $\hat{\lambda}_i(z)|dz| = \lambda_i(t)|dt|$, with $t = z^{\nu_k}$, in a deleted neighborhood of $z = 0$. Consider δ as a function of z: $\delta = \log \hat{\lambda}_{i+1}(z) - \log \hat{\lambda}_i(z)$. By Lemma 1, δ has neither positive maxima nor negative minima in the neighborhood considered. Thus δ has neither a positive maximum nor a negative minimum at P_k.

At a point P_k with $\nu_k = \infty$, we claim that δ is continuous and $\delta = 0$. In §III we showed that the asymptotic behavior of the Poincaré metric near a point P_k with $\nu_k = \infty$ is always given by $-1/(2|t| \log |t|)$, as $t \to 0$. (P_k corresponds to $t = 0$.) Thus $\delta \sim \log 1 = 0$, as $t \to 0$.

We therefore conclude that δ has neither positive maxima nor negative minima not only on S_i^b but also on S_i.

Next we claim that $\delta \leqq 0$ on S_i. For this purpose, we have to examine the behavior of δ on the boundary of S_i. The boundary of S_i consists of a finite number of simple closed analytic curves $C_1, \ldots, C_r$. They lie in the interior of S_{i+1} and do not pass through any of the points P_k, $k = 1, 2, 3, \ldots$. Let us consider U instead of Δ. Let Q be a point of C_j. Then a preimage of Q is a point $z_0 \in R$. In a neighborhood of z_0, the natural projection from U onto S_i can be expressed as $t = z - z_0$. Hence $\lambda_i(t) = (i/(z - \bar{z}))|dz/dt| \sim \infty$, as $t \to 0$. On the other hand, Q is an interior point of S_{i+1}, and $Q \neq P_k$, $k = 1, 2, 3, \ldots$, so that $\lambda_{i+1}(t)$ is defined at Q. It follows that $\delta \sim -\infty$, as $t \to 0$. A similar argument applies to all points of $C_1, C_2, \ldots, C_r$. It follows easily that $\delta \leqq 0$ on S_i, so that $\lambda_{i+1}(t) \leqq \lambda_i(t)$ on S_i^b.

With $S_i \subset\subset S$, we apply the same argument to the metrics $\lambda_i(t)|dt|$ and $\lambda(t)|dt|$ and conclude in exactly the same way that $\lambda(t) \leqq \lambda_i(t)$ on S_i^b.

(ii) It follows from part (i) that the metrics considered converge in the following sense: if D is a relatively compact domain in S^b with a local parameter t defined in D, then $\lim_{i \to \infty} \lambda_i(t) = \sigma(t)$ exists on D. (Note that $\lambda_i(t)$ will be defined for i sufficiently large, since $D \subset S_i^b$ for large i.) It is clear that $\sigma(t)|dt|$ is independent of the choice of local parameter.

We show next that the convergence $\lambda_i \to \sigma$ is normal. More precisely, if D is as before, and $D_0 \subset\subset D$, then $\lambda_i(t) \to \sigma(t)$ uniformly in D_0 as $i \to \infty$. From part (i), we have $\lambda(t) \leq \lambda_i(t) \leq \lambda_{i_0}(t)$ for all $i \geq i_0$, where i_0 is chosen such that λ_i, $i \geq i_0$, is defined in D. Thus $\log \lambda(t) \leq \log \lambda_i(t) \leq \log \lambda_{i_0}(t)$ in D, so that $|\log \lambda_i(t)| \leq M$ for $i \geq i_0$. By Lemma 2, we conclude that any infinite sequence of the family $\{\log \lambda_i\}$ has a uniformly convergent subsequence in D_0. The same is true for the family $\{\lambda_i\}$. But $\lambda_i \to \sigma$ in D as $i \to \infty$, it follows that the convergence is uniform in D_0.

Furthermore, by Lemma 3 we conclude that $\log \sigma(t)$ is of class C^2, hence $\sigma(t)$ is positive, of class C^2, and $\sigma(t)$ satisfies (1) in D. In conclusion, $\sigma(t)|dt|$ defines a metric on S^b, which has constant Gaussian curvature -4.

We shall show that near any point P_k with $2 \leq \nu_k < \infty$, $\sigma(t) \sim d|t|^{1/\nu_k - 1}$, as $t \to 0$, where d is a positive constant. (Here P_k corresponds to $t = 0$.) Let D be a relatively compact domain in S containing P_k in interior and with a local parameter t defined in it. (P_k corresponds to $t = 0$.) Also assume that D contains no other points P_k with $2 \leq \nu_k$. Then there exists i_0 such that, for $i \geq i_0$, λ_i is defined in $D - \{P_k\}$. By part (i), we have $\lambda(t) \leq \lambda_i(t) \leq \lambda_{i_0}(t)$ in $D - \{P_k\}$. Since $\lambda(t) \sim b|t|^{1/\nu_k - 1}$, $\lambda_i(t) \sim b_i|t|^{1/\nu_k - 1}$, as $t \to 0$, where b, b_i are positive constants and $i \geq i_0$, and $\lambda_i \to \sigma$ normally in $D - \{P_k\}$ as $i \to \infty$, we conclude from the corollary of Lemma 6 that $\sigma(t) \sim d|t|^{1/\nu_k - 1}$, as $t \to 0$, where d is a positive constant.

Finally, from part (i) and the fact that $\lambda_i \to \sigma$ as $i \to \infty$, we have $\lambda(t) \leq \sigma(t)$ on S^b. For any (sufficiently smooth) curve γ from a point of S other than P_k with $\nu_k = \infty$ to a point P_k with $\nu_k = \infty$ or to the ideal boundary of S, $\int_\gamma \lambda(t)|dt| = \infty$. Therefore $\int_\gamma \sigma(t)|dt| = \infty$. By Theorem 2, $\sigma(t) = \lambda(t)$ on S. This implies the conclusion of part (ii).

V. The monotonicity theorem for the metric.

DEFINITION 3. Given $\mathbf{v}' = \{\nu_k'\}$, $\mathbf{v}'' = \{\nu_k''\}$, we define $\mathbf{v}' \leq \mathbf{v}''$ if and only if $\nu_k' \leq \nu_k''$ for all k.

THEOREM 4. *Given $S, \mathbf{P}, \mathbf{v}', \mathbf{v}''$ such that $\mathbf{v}' \leq \mathbf{v}''$. Assume that the Riemann surfaces with signature $(S, \mathbf{P}, \mathbf{v}')$ and $(S, \mathbf{P}, \mathbf{v}'')$ are nonexceptional. Let $\lambda'(t)|dt|$, $\lambda''(t)|dt|$ be the Poincaré metrics on S, defined by the Fuchsian groups G', G'' which represent $(S, \mathbf{P}, \mathbf{v}')$, $(S, \mathbf{P}, \mathbf{v}'')$ respectively. Then $\lambda'(t) \leq \lambda''(t)$ on $S^* = S - \bigcup_{\nu_k' \text{ or } \nu_k'' \geq 2} \{P_k\}$. ($t$ is the same local parameter.)*

Proof. As in §IV, we choose a sequence $\{S_i\}$ of subdomains of S with the described properties. We may assume that $(S_i, \mathbf{P}, \mathbf{v}')$ and $(S_i, \mathbf{P}, \mathbf{v}'')$ are not exceptional

for $i = 1, 2, 3, \ldots$. We can represent $(S_i, \boldsymbol{P}, \boldsymbol{\nu}')$ by a Fuchsian group G_i' so that S_i carries a Poincaré metric defined by G_i'. Denote it by $\lambda_i'(t)|dt|$. Similarly, we can represent $(S_i, \boldsymbol{P}, \boldsymbol{\nu}'')$ by a Fuchsian group G_i'' so that S_i carries a Poincaré metric induced by G_i''. Denote it by $\lambda_i''(t)|dt|$.

By Theorem 3, $\lambda_i'(t) \to \lambda'(t)$ as $i \to \infty$ on $S - \bigcup_{\nu_k' \geq 2} \{P_k\}$ and $\lambda_i''(t) \to \lambda''(t)$ as $i \to \infty$ on $S - \bigcup_{\nu_k'' \geq 2} \{P_k\}$. We shall prove that $\lambda_i'(t) \leq \lambda_i''(t)$ on $S_i^* = S_i - \bigcup_{\nu_k' \text{ or } \nu_k'' \geq 2} \{P_k\}$; this will imply the assertion. On S_i^* set $\delta(t) = \log \lambda_i'(t) - \log \lambda_i''(t)$. Then as in §IV, we can show that δ is a function defined on S_i^* and has neither positive maxima nor negative minima there. To show $\delta \leq 0$ on S_i^*, we shall study the behavior of δ at the points P_k with either ν_k' or $\nu_k'' \geq 2$ on S_i and also on the boundary of S_i.

Case (i). At a point P_k with $2 \leq \nu_k' = \nu_k'' < \infty$, δ is continuous and has neither a positive maximum nor a negative minimum.

Case (ii). At a point P_k with $\nu_k' = \nu_k'' = \infty$, δ is continuous and $\delta = 0$ there.

Case (iii). At a point P_k with $\nu_k' \neq \nu_k''$ (hence $1 \leq \nu_k' < \nu_k''$), we claim that $\delta \sim -\infty$ as $t \to 0$, where $t = 0$ corresponds to P_k. Indeed, if $\nu_k' = 1$, then $\lambda_i'(t)$ is continuous at $t = 0$. And $\lambda_i''(t) \sim b''|t|^{1/\nu_k'' - 1}$ or $\lambda_i''(t) \sim -1/(2|t| \log |t|)$, as $t \to 0$ according as $\nu_k'' < \infty$ or $\nu_k'' = \infty$, where b'' is a positive constant. In both cases, $\delta \sim -\infty$ as $t \to 0$. If $2 \leq \nu_k' < \nu_k'' < \infty$, we have $\lambda_i'(t) \sim b'|t|^{1/\nu_k' - 1}$, and $\lambda_i''(t) \sim b''|t|^{1/\nu_k'' - 1}$, as $t \to 0$, where b', b'' are positive constants, so that $\delta \sim -\infty$ as $t \to 0$. Finally, if $2 \leq \nu_k' < \nu_k'' = \infty$, we have $\lambda_i'(t) \sim b'|t|^{1/\nu_k' - 1}$, $\lambda_i''(t) \sim -1/(2|t| \log |t|)$, as $t \to 0$. By direct calculation, it follows that $\delta \sim -\infty$ as $t \to 0$.

Case (iv). On the boundary of S_i, we claim that δ is continuous and $\delta = 0$. As before, denote the boundary curves by $C_1, \ldots, C_r$. Let Q be a point on C_j. Consider U instead of Δ. $\lambda_i'(t)|dt|$, $\lambda_i''(t)|dt|$ come from the Poincaré metric in U via two different natural projections. Let $z_0 \in \boldsymbol{R}$ be a preimage of Q in the complex z-plane under one of the natural projections. Let $w_0 \in \boldsymbol{R}$ be a preimage of Q in the complex w-plane under the other natural projection.

There exists a conformal mappings of a (half) neighborhood of w_0 in the upper half w-plane onto a (half) neighborhood of z_0 in the upper half z-plane. Furthermore, it sends a segment of $\boldsymbol{R}$ onto $\boldsymbol{R}$. By the reflection principle, it can be extended to a conformal mapping of a full neighborhood of w_0 onto a full neighborhood of z_0. Expressing it in power series, we have $z - z_0 = a_1(w - w_0) + a_2(w - w_0)^2 + \cdots$, where $a_1 > 0$, and all coefficients are real.

On the other hand, $\lambda_i'(t)|dt| = (i/(z - \bar{z}))|dz|$, and $\lambda_i''(t)|dt| = (i/(w - \bar{w}))|dw|$, so that $\lambda_i'(t)/\lambda_i''(t) \sim (1/a_1)a_1 = 1$, as $t \to 0$. Thus $\delta = 0$ at Q. Similar considerations show that $\delta = 0$ on all curves $C_1, \ldots, C_r$.

It follows easily that $\delta \leq 0$ on S_i^*, so that $\lambda_i'(t) \leq \lambda_i''(t)$ on S_i^*.

VI. The limit theorem for the metric.

DEFINITION 4. Given $\boldsymbol{\nu}^j = \{\nu_k^j\}$, $j = 1, 2, 3, \ldots$, and $\boldsymbol{\nu} = \{\nu_k\}$, we define $\boldsymbol{\nu}^j \to \boldsymbol{\nu}$, as $j \to \infty$, to mean that $\nu_k^j \to \nu_k$, as $j \to \infty$, for all k.

Theorem 5. *Given* $S, P, \nu^0, \nu, \nu^j, j = 1, 2, 3, \ldots,$ *such that* $\nu^0 \leqq \nu^j$ *for all* j *and* $\nu^j \to \nu,$ *as* $j \to \infty.$ *Assume that* $(S, P, \nu^0), (S, P, \nu), (S, P, \nu^j), j = 1, 2, 3, \ldots,$ *are all non-exceptional Riemann surfaces with signature. Let* $\lambda_0(t)|dt|,$ $\lambda(t)|dt|,$ $\lambda_j(t)|dt|,$ $j = 1, 2, 3, \ldots,$ *be the Poincaré metrics on* $S,$ *induced by the Fuchsian groups* $G_0, G, G_j,$ $j = 1, 2, 3, \ldots,$ *which represent* $(S, P, \nu^0), (S, P, \nu), (S, P, \nu^j), j = 1, 2, 3, \ldots,$ *respectively.*

Then $\lambda_j(t) \to \lambda(t)$ *normally on* $S^b = S - \bigcup_{\nu_k \geqq 2} \{P_k\},$ *as* $j \to \infty.$ (*Here* t *is the same local parameter for all* λ_j *and* $\lambda.$)

Proof. Define $\nu^\infty = \{\nu_k^\infty\}$ with $\nu_k^\infty = \infty$ for all $k.$ (S, P, ν^∞) is then a nonexceptional Riemann surface with signature. Let $\lambda_\infty(t)|dt|$ be the Poincaré metric on $S,$ defined by the Fuchsian group G_∞ which represents $(S, P, \nu^\infty).$ Let D be a parametric disk in $S_0 = S - \{$all points $P_k\}$ with local parameter $t.$ Since $\nu^0 \leqq \nu^j \leqq \nu^\infty,$ by Theorem 4, $\lambda_0(t) \leqq \lambda_j(t) \leqq \lambda_\infty(t),$ hence $\log \lambda_0(t) \leqq \log \lambda_j(t) \leqq \log \lambda_\infty(t)$ in $D.$ Let $D_0 \subset\subset D_1 \subset\subset D.$ Then $|\log \lambda_j(t)| \leqq M$ in $D_1,$ for all $j.$ By Lemma 2, any infinite sequence of the family $\{\log \lambda_j\},$ $j = 1, 2, 3, \ldots,$ has a uniformly convergent subsequence in $D_0.$ The same is true for the family $\{\lambda_j\},$ $j = 1, 2, 3, \ldots.$

Consider a point $P = P_k$ with $\nu_k = 1.$ $P \in S^b.$ (But $P \notin S_0.$) Let D be a parametric disk in S^b with local parameter t containing P and no other points P_k ($t = 0$ at P). Let $P \in D_0 \subset\subset D_1 \subset\subset D.$ By assumption, $\nu_k^j \to \nu_k = 1$ as $j \to \infty,$ hence there exists j_0 such that $\nu_k^j = 1$ for $j \geqq j_0.$ Define $\nu' = \{\nu_n'\}$ with $\nu_n' = \infty$ for $n \neq k, \nu_n' = 1$ for $n = k.$ Then (S, P, ν') is a nonexceptional Riemann surface with signature. Let $\lambda'(t)|dt|$ be the Poincaré metric on $S,$ defined by the Fuchsian group G' which represents $(S, P, \nu').$ Since $\nu^0 \leqq \nu^j \leqq \nu',$ for $j \geqq j_0,$ by Theorem 4, $\lambda_0(t) \leqq \lambda_j(t) \leqq \lambda'(t),$ hence $\log \lambda_0(t) \leqq \log \lambda_j(t) \leqq \log \lambda'(t)$ in $D,$ so that $|\log \lambda_j(t)| \leqq M$ in $D_1.$ By Lemma 2 again, any infinite sequence of the family $\{\log \lambda_j\}, j \geqq j_0,$ has a uniformly convergent subsequence in $D_0.$ The same is clearly true for the family $\{\log \lambda_j\}, j = 1, 2, 3, \ldots,$ hence for the family $\{\lambda_j\}, j = 1, 2, 3, \ldots,$ also.

Since every Riemann surface has a countable base, S^b can be covered by a countable number of domains like D_0 described above. By the usual diagonal method, one sees that, on $S^b,$ any infinite sequence of the family $\{\lambda_j\}, j = 1, 2, 3, \ldots,$ has a normally convergent subsequence. Let $\{\lambda_{j_k}\}$ be such a subsequence and $\lambda_{j_k}(t) \to \sigma(t)$ as $j_k \to \infty.$ We shall prove that $\sigma(t) = \lambda(t)$ on $S^b.$ This will imply that the original sequence $\{\lambda_j\}, j = 1, 2, 3, \ldots,$ converges to λ normally on $S^b,$ as $j \to \infty.$ For simplicity, we may assume that $\lambda_j(t) \to \sigma(t)$ normally on $S^b,$ as $j \to \infty,$ and show that $\sigma(t) = \lambda(t).$

By Lemma 3, we conclude that $\log \sigma(t)$ is of class $C^2,$ hence $\sigma(t)$ is positive and of class $C^2,$ and $\sigma(t)$ satisfies (1) in $D_0.$ Clearly $\sigma(t)|dt|$ is invariant, so that it defines a metric on $S^b.$ It has constant Gaussian curvature -4 on $S^b.$ (Compare §IV.) We shall show that near any point P_k with $2 \leqq \nu_k < \infty,$ $\sigma(t) \sim d|t|^{1/\nu_k - 1},$ as $t \to 0,$ where d is a positive constant. (Here P_k corresponds to $t = 0.$) Let D be a parametric disk in S with local parameter $t,$ containing P_k (P_k corresponds to $t = 0$)

and no other points P_j, $j \neq k$, in its interior. By assumption, $\nu_k^j \to \nu_k$ as $j \to \infty$. Since $\nu_k < \infty$, there exists j_0 such that $\nu_k^j = \nu_k$ for $j \geqq j_0$. Define $\mathbf{v}'' = \{\nu_n''\}$ with $\nu_n'' = \infty$ for $n \neq k$, $\nu_n'' = \nu_k$ for $n = k$. Then $(S, \mathbf{P}, \mathbf{v}'')$ is a nonexceptional Riemann surface with signature. Let $\lambda''(t)|dt|$ be the Poincaré metric on S, defined by the Fuchsian group G'' which represents $(S, \mathbf{P}, \mathbf{v}'')$. Since $\mathbf{v}^0 \leqq \mathbf{v}^j \leqq \mathbf{v}''$, for $j \geqq j_0$, by Theorem 4, $\lambda_0(t) \leqq \lambda_j(t) \leqq \lambda''(t)$ in $D - \{P_k\}$. Since $\lambda_j \to \sigma$ normally in $D - \{P_k\}$ as $j \to \infty$, we conclude from the corollary of Lemma 6 that $\sigma(t) \sim d|t|^{1/\nu_k - 1}$ as $t \to 0$, where d is a positive constant.

Finally, we shall show that if γ is a (sufficiently smooth) curve on S beginning at a point P other than P_k with $\nu_k = \infty$ and leading to a point P_k with $\nu_k = \infty$ or to the ideal boundary of S, then $\int_\gamma \sigma(t)|dt| = \infty$. The latter case follows from the fact that $\lambda_0(t) \leqq \lambda_j(t)$ for every j, so that $\lambda_0(t) \leqq \sigma(t)$.

In the former case, let P_k be a point with $\nu_k = \infty$. By assumption, $\nu_k^j \to \infty$, as $j \to \infty$. Let D be a parametric disk in S with local parameter t, containing P_k (P_k corresponds to $t = 0$) and no other points P_j, $j \neq k$, in its interior. In D, draw a circle L: $|t| = R$. We may assume that $R > 1$. Also, we may assume that γ is a (sufficiently smooth) curve from a point on $|t| = 1$ to $t = 0$, lying completely in $|t| \leqq 1$. Let us consider a fixed j and let us write n instead of ν_k^j. Assume first that $n < \infty$. By the corollary of Lemma 5, there exists a function $\hat{\lambda}(z)$, of class C^2, defined for $|z| < R^{1/n}$, satisfying (1) and such that $\lambda_j(t)|dt| = \hat{\lambda}(z)|dz|$, with $t = z^n$. By assumption, $\mathbf{v}^0 \leqq \mathbf{v}^j$, for all j, hence $\lambda_0(t) \leqq \lambda_j(t)$ in $D - \{P_k\}$, so that $c < \lambda_j(t)$ on $|t| = 1$ for all j, where c is a positive constant independent of j. It follows that on $|z| = 1$, $cn < \hat{\lambda}(z)$.

We introduce a metric $\tau(z)|dz|$ with
$$\tau(z) = a/(1 - a^2|z|^2), \qquad |z| < R',$$
where $0 < a < 1$ is a constant chosen so that $a/(1 - a^2) = cn$, that is,
$$a = ((4c^2n^2 + 1)^{1/2} - 1)/2cn, \quad \text{and} \quad R' = \min(R^{1/n}, 1/a).$$
Note that $R' > 1$ and $a \to 1$ as $n \to \infty$. Under this choice of a, $\tau(z) = cn < \lambda(z)$ on $|z| = 1$. Set $\delta = \log \tau(z) - \log \hat{\lambda}(z)$. Since $\tau(z)$ satisfies (1), we have, by Lemma 1, that δ has neither positive maxima nor negative minima in $|z| < R'$. But $\delta < 0$ on $|z| = 1$ by construction. Thus $\delta \leqq 0$ for $|z| \leqq 1$ and we conclude that $\tau(z) \leqq \hat{\lambda}(z)$ for $|z| \leqq 1$.

Observe that $\tau(z)|dz|$ is a Poincaré metric in the disk $|z| < 1/a$. Hence in $|z| \leqq 1$, any geodesic through the point $z = 0$ with respect to this metric is a straight line and vice versa. Also, any diameter has the same length in this metric. Now let ε be a small positive number and let γ_ε be the arc of γ leading from the initial point of γ to the first point on the intersection of γ and the circle $|t| = \varepsilon$. The arc γ_ε is the image of an arc $\hat{\gamma}_\varepsilon$ under the mapping $z \mapsto z^n$. The arc $\hat{\gamma}_\varepsilon$ leads from a point on $|z| = 1$ to a point on $|z| = \varepsilon^{1/n}$. We have

$$\int_{\gamma_\varepsilon} \lambda_j(t)|dt| = \int_{\hat{\gamma}_\varepsilon} \hat{\lambda}(z)|dz| \geqq \int_{\hat{\gamma}_\varepsilon} \tau(z)|dz|.$$

But $\int_{\hat{\gamma}_\varepsilon} \tau(z)|dz|$ is not less than the distance from the circle $|z|=1$ to the circle $|z|=\varepsilon^{1/n}$ in the metric $\tau(z)|dz|$, that is,

$$\int_{\hat{\gamma}_\varepsilon} \tau(z)|dz| \geqq \int_{\varepsilon^{1/n}}^1 \frac{a}{1-a^2x^2}\, dx$$

$$= \frac{1}{2} \log \left(\frac{1+a}{1+a\varepsilon^{1/n}} \frac{1-a\varepsilon^{1/n}}{1-a}\right) > \frac{1}{2} \log \frac{1-a\varepsilon^{1/n}}{1-a},$$

and therefore

$$(3) \qquad\qquad\qquad \int_{\gamma_\varepsilon} \lambda_j(t)|dt| > \frac{1}{2} \log N_n,$$

where $N_n = (1-a\varepsilon^{1/n})/(1-a)$, $a = ((4c^2n^2+1)^{1/2}-1)/2cn$. We have proven this inequality under the assumption that $\nu_k^j = n < \infty$. We remark now that if $\nu_k^j = \infty$, then it holds for every n. This follows at once from Theorem 4.

Next, for $n \to \infty$ we have

$$a = (1+1/4c^2n^2)^{1/2} - 1/2cn = 1-1/2cn + O(1/n^2),$$
$$\varepsilon^{1/n} = 1 + (\log \varepsilon)/n + O(1/n^2),$$

so that

$$N_n = (1-a\varepsilon^{1/n})/(1-a) = 1 + 2c \log(1/\varepsilon) + O(1/n).$$

Thus by (3),

$$\liminf_{j \to \infty} \int_{\gamma_\varepsilon} \lambda_j(t)|dt| > \log(1+2c \log(1/\varepsilon))^{1/2}.$$

Hence

$$\int_{\gamma_\varepsilon} \sigma(t)|dt| > \log(1+2c \log(1/\varepsilon))^{1/2}$$

and

$$\int_{\gamma} \sigma(t)|dt| = \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \sigma(t)|dt| = \infty,$$

as asserted.

By Theorem 2, we conclude that $\sigma(t) = \lambda(t)$ on S. The proof of Theorem 5 is complete.

VII. **The limit theorem for Fuchsian groups.** As above, let S, P, ν, ν^j, $j = 1, 2, 3$, ..., be given such that the Riemann surfaces with signature (S, P, ν), (S, P, ν^j), $j = 1, 2, 3, \ldots$, are nonexceptional. Then they can be represented by Fuchsian groups G, G_j, $j = 1, 2, 3, \ldots$, with natural projections g, g_j, $j = 1, 2, 3, \ldots$, from the unit disk onto the respective Riemann surfaces with signature. For the sake of uniformity in notation, we shall write G_∞, g_∞ for G, g respectively. We shall

normalize g_∞, g_j in the following manner. Let P be a point fixed in S_0, and let a direction v at P be fixed. We normalize g_∞, g_j by the condition that the origin 0 of the unit disk is mapped onto P and the direction of the positive real axis onto the direction v. Then g_∞, g_j, G_∞, G_j are uniquely determined. Note that g_∞: $\Delta_\infty \to S_0$, g_j: $\Delta_j \to S_0$, where $\Delta_\infty = \Delta - \{$all the preimages of P_k, $1 \le \nu_k < \infty$, under $g_\infty\}$, and Δ_j is defined similarly.

Construct the universal covering Δ of Δ_∞ and normalize the projection h_∞ by the conditions that $h_\infty(0) = 0$, $h'_\infty(0) > 0$. Also let $f_\infty = g_\infty \circ h_\infty$, then f_∞ is the universal covering projection of S_0 with the property that the origin $0 \in \Delta$ is mapped onto P and the direction of the positive real axis to the direction v. Similarly, construct the universal covering Δ of Δ_j, normalize the projection h_j by the conditions that $h_j(0) = 0$, $h'(0) > 0$, and let $f_j = g_j \circ h_j$, then f_j is the universal covering projection of S_0 with the property that the origin $0 \in \Delta$ is mapped onto P and the direction of the positive real axis to the direction v. It follows that $f_\infty = f_j$. From now on, we simply write f for f_∞, f_j.

Let F be the covering group of f, H_∞ the covering group of h_∞, H_j the covering group of h_j. It is known that we have the following exact sequences of groups and homomorphisms:

$$1 \longrightarrow H_\infty \hookrightarrow F \xrightarrow{\chi_\infty} G_\infty \longrightarrow 1, \qquad 1 \longrightarrow H_j \hookrightarrow F \xrightarrow{\chi_j} G_j \longrightarrow 1,$$

such that for every $\alpha \in F$, $h_\infty \circ \alpha = \chi_\infty(\alpha) \circ h_\infty$, $h_j \circ \alpha = \chi_j(\alpha) \circ h_j$. (See e.g. Bers [4].)

THEOREM 6. *Given S, P, ν^0, ν, ν^j, $j = 1, 2, 3, \ldots$, such that $\nu^0 \le \nu^j$ for all j and $\nu^j \to \nu$ as $j \to \infty$. Assume that (S, P, ν^0), (S, P, ν), (S, P, ν^j), $j = 1, 2, 3, \ldots$, are all nonexceptional Riemann surfaces with signature. Define h_j, g_j, h_∞, g_∞, f, H_j, G_j, H_∞, G_∞, F, χ_j, χ_∞ as above.*

Then $h_j \to h_\infty$ and, for every $\alpha \in F$, $\chi_j(\alpha) \to \chi_\infty(\alpha)$ normally on Δ as $j \to \infty$.

Proof. (a) We shall first prove that $h_j(z) \to h_\infty(z)$ normally on Δ as $j \to \infty$. Note that $|h_j(z)| < 1$, $z \in \Delta$, for every j. Hence $\{h_j\}$ is a normal family of holomorphic functions. That is, in every infinite sequence of this family, there exists a normally convergent subsequence, say, with limit function h. h is holomorphic in Δ. If we can prove that $h = h_\infty$, then the original sequence $\{h_j\}$ converges to h_∞ normally on Δ as $j \to \infty$. We may assume that $h_j \to h$ normally on Δ as $j \to \infty$. We shall prove that $h = h_\infty$. By assumption, $h_j(0) = 0$, $h'_j(0) > 0$. Therefore $h(0) = 0$, $h'(0) \ge 0$. We shall show that in fact $h'(0) > 0$.

Let $\lambda_j(t)|dt|$, $\lambda_\infty(t)|dt|$ be the Poincaré metrics on S defined by G_j, G_∞. By Theorem 5, $\lambda_j(t) \to \lambda_\infty(t)$ on S_0 as $j \to \infty$. Lift, by f, the metrics $\lambda_j(t)|dt|$, $\lambda_\infty(t)|dt|$ on S_0 to metrics $\rho_j(z)|dz|$, $\rho_\infty(z)|dz|$ on Δ. Hence $\rho_j(z)|dz| = \lambda_j(t)|dt|$, $\rho_\infty(z)|dz| = \lambda_\infty(t)|dt|$, with $t = f(z)$, and $\rho_j(z) \to \rho_\infty(z)$ on Δ as $j \to \infty$. On the other hand, if we lift the metrics $\lambda_j(t)|dt|$ on S_0 to Δ_j by g_j and $\lambda_\infty(t)|dt|$ on S_0 to Δ_∞ by g_∞, we clearly get back the Poincaré metric on the unit disk. Therefore,

$$\rho_j(z) = |h'_j(z)| / (1 - |h_j(z)|^2), \qquad \rho_\infty(z) = |h'_\infty(z)| / (1 - |h_\infty(z)|^2)$$

for $z \in \Delta$. Note that $h_j \to h$, $h'_j \to h'$ and $\rho_j \to \rho_\infty$ as $j \to \infty$, so that

$$\rho_\infty(z) = |h'(z)|/(1 - |h(z)|^2)$$

for $z \in \Delta$. Since $\rho_\infty(0) > 0$, it follows that $h'(0) \neq 0$, hence $h'(0) > 0$.

Altogether,

(*) $h(0) = 0$, $h'(0) > 0$,   and $|h'(z)| = \rho_\infty(z)(1 - |h(z)|^2)$,   for $z \in \Delta$.

We claim that these three conditions determine $h(z)$ uniquely. It suffices to show that they determine all derivatives $h^{(n)}(0)$, $n = 1, 2, \ldots$. Square (*) and put $K(z, \bar{z}) = \rho_\infty^2(z)$. We can show that $|h'(0)|^2 = K(0, 0)$. By assumption $h'(0) > 0$, therefore $h'(0)$ is completely determined. Further differentiation gives that $h^{(n)}(0)(h'(0))^{-} = \partial^{n-1}K(0, 0)/\partial z^{n-1}$. Therefore, $h^{(n)}(0)$ is completely determined for all n. It follows immediately that $h = h_\infty$, as asserted.

(b) We shall prove that for every $\alpha \in F$, $\chi_j(\alpha) \to \chi_\infty(\alpha)$ normally on Δ as $j \to \infty$. For simplicity, write α_j for $\chi_j(\alpha)$, α_∞ for $\chi_\infty(\alpha)$. Recall that $\alpha_j \in G_j$, $\alpha_\infty \in G_\infty$ are Möbius transformations from Δ onto itself and $h_j(\alpha(z)) = \alpha_j(h_j(z))$, $h_\infty(\alpha(z)) = \alpha_\infty(h_\infty(z))$, for $z \in \Delta$. By part (a), we conclude that $\alpha_j(h_j(z)) \to \alpha_\infty(h_\infty(z))$, $z \in \Delta$, as $j \to \infty$. From this we shall show that $\alpha_j \to \alpha_\infty$ normally on Δ as $j \to \infty$.

Now $|\alpha_j(z)| < 1$ for all j. It follows that $\{\alpha_j\}$ is a normal family of holomorphic functions, that is, any infinite sequence in the family has a normally convergent subsequence. If we can prove that the limit function is always α_∞, then $\alpha_j \to \alpha_\infty$ normally on Δ as $j \to \infty$. As usual, we may assume that $\alpha_j \to \hat{\alpha}$ normally on Δ as $j \to \infty$, and we shall prove that $\hat{\alpha} = \alpha_\infty$.

Let z_i, $i = 1, 2, 3$, be three distinct points in Δ such that the points $a_i = h_\infty(z_i)$, $i = 1, 2, 3$, are also distinct. Let $b_i = \alpha_\infty(a_i) = \alpha_\infty(h_\infty(z_i))$, $i = 1, 2, 3$. Thus α_∞ maps a_i onto b_i, $i = 1, 2, 3$, respectively. On the other hand, let $a_i^j = h_j(z_i)$, $i = 1, 2, 3$. Then by part (a), $a_i^j \to a_i$ as $j \to \infty$, $i = 1, 2, 3$. Let $b_i^j = \alpha_j(a_i^j) = \alpha_j(h_j(z_i))$, $i = 1, 2, 3$. From the fact that $\alpha_j(h_j(z)) \to \alpha_\infty(h_\infty(z))$ as $j \to \infty$, we conclude that $b_i^j \to b_i$ as $j \to \infty$, $i = 1, 2, 3$. The fact that a_i, $i = 1, 2, 3$, are distinct implies that a_i^j, $i = 1, 2, 3$, are also distinct for j large. Therefore, for j large, $w = \alpha_j(z)$ can be expressed as

$$\frac{(w - b_1^j)(b_2^j - b_3^j)}{(w - b_3^j)(b_2^j - b_1^j)} = \frac{(z - a_1^j)(a_2^j - a_3^j)}{(z - a_3^j)(a_2^j - a_1^j)}.$$

It follows that the limit $\hat{\alpha}$ of α_j, $w = \hat{\alpha}(z)$, can be expressed as

$$\frac{(w - b_1)(b_2 - b_3)}{(w - b_3)(b_2 - b_1)} = \frac{(z - a_1)(a_2 - a_3)}{(z - a_3)(a_2 - a_1)}.$$

Therefore $\hat{\alpha}$ is a Möbius transformation mapping the distinct points a_i onto b_i, $i = 1, 2, 3$, respectively. It follows that $\hat{\alpha} = \alpha_\infty$.

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